Math 150, *Lecture Notes- Bonds* Name____________

Section 2.5 *Implicit Differentiation*

Implicit and Explicit Functions

Up to this point in the text, most functions have been expressed in **explicit form.** For example, in the equation

$y = 3x^2 - 5$ Explicit form

the variable y is explicitly written as a function of x. Some functions, however, are only implied by an equation. For instance, the function $y = 1/x$ is defined **implicitly** by the equation $xy = 1$. Suppose you were asked to find dy/dx for this equation. You could begin by writing y explicitly as a function of x and then differentiating.

Implicit Form	*Explicit Form*	*Derivative*
$xy = 1$	$y = \dfrac{1}{x} = x^{-1}$	$\dfrac{dy}{dx} = -x^{-2} = -\dfrac{1}{x^2}$

This strategy works whenever you can solve for the function explicitly. You cannot, however, use this procedure when you are unable to solve for y as a function of x. For instance, how would you find dy/dx for the equation

$x^2 - 2y^3 + 4y = 2$

where it is very difficult to express y as a function of x explicitly? To do this, you can use **implicit differentiation.**

To understand how to find dy/dx implicitly, you must realize that the differentiation is taking place *with respect to x*. This means that when you differentiate terms involving x alone, you can differentiate as usual. However, when you differentiate terms involving y, you must apply the Chain Rule, because you are assuming that y is defined implicitly as a differentiable function of x.

Ex.1 Differentiating with Respect to x.

a. $\dfrac{d}{dx}[x^3] = 3x^2$ Variables agree: use Simple Power Rule.

Variables agree

b. $\dfrac{d}{dx}[\underbrace{y^3}_{u^n}] = \underbrace{3y^2}_{nu^{n-1}}\underbrace{\dfrac{dy}{dx}}_{u'}$ Variables disagree: use Chain Rule.

Variables disagree

c. $\dfrac{d}{dx}[x + 3y] = 1 + 3\dfrac{dy}{dx}$ Chain Rule: $\dfrac{d}{dx}[3y] = 3y'$

d. $\dfrac{d}{dx}[xy^2] = x\dfrac{d}{dx}[y^2] + y^2\dfrac{d}{dx}[x]$ Product Rule

$= x\left(2y\dfrac{dy}{dx}\right) + y^2(1)$ Chain Rule

$= 2xy\dfrac{dy}{dx} + y^2$ Simplify.

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$\frac{d}{dx}(y) = \frac{dy}{dx}$

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Variables agree

outside Inside

$y = y(x)$

b. $\frac{d}{dx}[\underbrace{y^3}_{u^n}] = \underbrace{3y^2}_{nu^{n-1}} \underbrace{\frac{dy}{dx}}_{u'}$ Variables disagree: use Chain Rule.

Leibniz Rule

Variables disagree

$= \frac{d}{dx}(x) + 3 \cdot \frac{d}{dx}(y)$

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Guidelines for Implicit Differentiation

1. Differentiate both sides of the equation *with respect to x*.
2. Collect all terms involving dy/dx on the left side of the equation and move all other terms to the right side of the equation.
3. Factor dy/dx out of the left side of the equation.
4. Solve for dy/dx.

Ex.2 Find $\dfrac{dy}{dx}$, given that $2x^3+3y^3=64$.

$$\frac{d}{dx}\left[2x^3+3y^3\right]=\frac{d}{dx}[64]$$

$$2\cdot\frac{d}{dx}[x^3]+3\cdot\frac{d}{dx}[y^3]=0$$

Implicit Differentiation

$$2\cdot 3x^2+3\cdot 3y^2\cdot\frac{dy}{dx}=0$$

Chain Rule

$$6x^2+9y^2\cdot\frac{dy}{dx}=0$$

$$9y^2\frac{dy}{dx}=-6x^2$$

$$\frac{dy}{dx}=\frac{-6x^2}{9y^2}$$

$$\frac{dy}{dx}=-\frac{2x^2}{3y^2}$$

Ex.3 Find $\frac{dy}{dx}$, given that $x^2y+y^2x=-2$.

$$\frac{d}{dx}(x^2y+y^2x)=\frac{d}{dx}(2)$$

$$\frac{d}{dx}(x^2y)+\frac{d}{dx}(y^2x)=0$$

$$x^2\cdot\frac{d}{dx}(y)+y\cdot\frac{d}{dx}(x^2)+y^2\cdot\frac{d}{dx}(x)+x\cdot\frac{d}{dx}(y^2)=0$$

$$x^2\frac{dy}{dx}+y\cdot(2x)+y^2\cdot 1+x\cdot 2y\cdot\frac{dy}{dx}=0$$

$$x^2\cdot\frac{dy}{dx}+2xy+y^2+2xy\cdot\frac{dy}{dx}=0$$

$$x^2\cdot\frac{dy}{dx}+2xy\cdot\frac{dy}{dx}=-y^2-2xy$$

$$\frac{dy}{dx}\left[x^2+2xy\right]=-(y^2+2xy)$$

$$\frac{dy}{dx}=\frac{-(y^2+2xy)}{x^2+2xy}$$

Ex.4 Find $\frac{dy}{dx}$ and evaluate the derivative at $(2,2)$, given that $y^3-x^2=4$.

$$\frac{d}{dx}\left[y^3-x^2\right]=\frac{d}{dx}[4]$$

$$\frac{d}{dx}[y^3]-\frac{d}{dx}[x^2]=0$$

$$3y^2\cdot\frac{dy}{dx}-2x=0$$

$$3y^2\cdot\frac{dy}{dx}=2x$$

$$\frac{dy}{dx}=\frac{2x}{3y^2}$$

Tells us the slope of tangent lines

$$\left.\frac{dy}{dx}\right|_{(2,2)}=\frac{2(2)}{3(2)^2}$$

$$\left.m_{Tan}\right|_{(2,2)}=\frac{\cancel{2}\cdot\cancel{2}}{\cancel{2}\cdot\cancel{2}\cdot 3}$$

$$\left.m_{Tan}\right|_{(2,2)}=\frac{1}{3}=\left.\frac{dy}{dx}\right|_{(2,2)}$$

$$y-y_1=m_{Tan}(x-x_1)$$

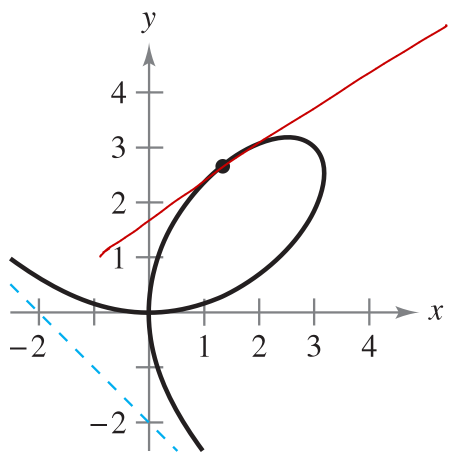
$$y-(2)=\frac{1}{3}\left[x-(2)\right]$$

$$2+y-2=\frac{1}{3}x-\frac{2}{3}+\frac{6}{3}$$

$$\boxed{y=\frac{1}{3}x+\frac{4}{3}}$$

$$y - y_1 = m_{TAN}(x - x_1)$$

Ex.5 Find the equation of the tangent line to the graph of $x^3 + y^3 - 6xy = 0$ at $\left(\frac{4}{3}, \frac{8}{3}\right)$. $= (x_1, y_1)$



$$\frac{d}{dx}\left[x^3 + y^3 - 6xy\right] = \frac{d}{dx}[0]$$

$$\frac{d}{dx}[x^3] + \frac{d}{dx}[y^3] - 6 \cdot \frac{d}{dx}[xy] = 0$$

$$3x^2 + 3y^2 \cdot \frac{dy}{dx} - 6\left[x \cdot \frac{d}{dx}(y) + y \cdot \frac{d}{dx}(x)\right] = 0$$

$$3x^2 + 3y^2 \cdot \frac{dy}{dx} - 6x \cdot \frac{dy}{dx} - 6y = 0$$

$$3y^2 \cdot \frac{dy}{dx} - 6x \cdot \frac{dy}{dx} = 6y - 3x^2$$

$$\frac{\frac{dy}{dx}\left[3y^2 - 6x\right]}{3y^2 - 6x} = \frac{6y - 3x^2}{3y^2 - 6x}$$

$$\frac{dy}{dx} = \frac{3 \cdot [2y - x^2]}{3[y^2 - 2x]}$$

$$\boxed{\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}}$$

$$\left.\frac{dy}{dx}\right|_{\left(\frac{4}{3}, \frac{8}{3}\right)} = \frac{2\left(\frac{8}{3}\right) - \left(\frac{4}{3}\right)^2}{\left(\frac{8}{3}\right)^2 - 2\left(\frac{4}{3}\right)}$$

$$= \left[\frac{\frac{16}{3} - \frac{16}{9}}{\frac{64}{9} - \frac{8}{3}}\right]\left[\frac{\frac{9}{1}}{\frac{9}{1}}\right]$$

$$= \frac{3 \cdot 16 - 16}{64 - 8 \cdot 3}$$

$$= \frac{48 - 16}{64 - 24}$$

$$= \frac{32}{40}$$

$$= \frac{8 \cdot 4}{8 \cdot 5}$$

$$= \frac{4}{5}$$

$$y - y_1 = m_{TAN}(x - x_1)$$

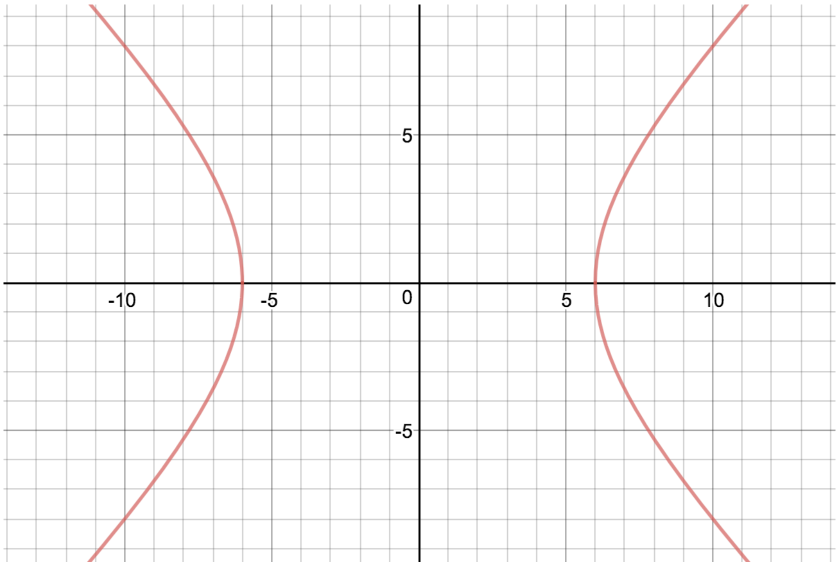
$$y - \left(\frac{8}{3}\right) = \left(\frac{4}{5}\right)\left[x - \left(\frac{4}{3}\right)\right]$$

$$y - \frac{8}{3} = \frac{4}{5}x - \frac{16}{15}$$

$$y - \frac{8}{3} + \frac{8}{3} = \frac{4}{5}x - \frac{16}{15} + \frac{40}{15}$$

$$\boxed{y = \frac{4}{5}x + \frac{24}{15}}$$

Ex.6 Find $\dfrac{d^2y}{dx^2}$, given that $x^2 - y^2 = 36$.



$$\frac{d}{dx}\left[x^2 - y^2\right] = \frac{d}{dx}\left[36\right]$$

$$\frac{d}{dx}(x^2) - \frac{d}{dx}(y^2) = 0$$

$$2x - 2y \cdot \frac{dy}{dx} = 0$$

$$2x = 2y \cdot \frac{dy}{dx}$$

$$\frac{2x}{2y} = \frac{dy}{dx}$$

$$\frac{x}{y} = \frac{dy}{dx}$$

$$\frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}\left[\frac{x}{y}\right]$$

$$\frac{d^2y}{dx^2} = \frac{y \cdot \frac{d}{dx}(x) - x \cdot \frac{d}{dx}(y)}{y^2}$$

$$\frac{d^2y}{dx^2} = \frac{y \cdot 1 - x \cdot \frac{dy}{dx}}{y^2}$$

$$\frac{d^2y}{dx^2} = \frac{y - x \cdot \left(\frac{x}{y}\right)}{y^2}$$

$$\frac{d^2y}{dx^2} = \left[\frac{\frac{y}{1} - \frac{x^2}{y}}{y^2}\right]\left[\frac{\frac{y}{1}}{\frac{y}{1}}\right]$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{y^2 - x^2}{y^3}}$$

$\frac{dy}{dx}$

@ $(-1,1)$

#25 $(x+y)^3 = x^3 + y^3$

$$\frac{d}{dx}\left[(x+y)^3\right] = \frac{d}{dx}\left[x^3 + y^3\right]$$

$$\underbrace{3(x+y)^2 \cdot \frac{d}{dx}(x+y)}_{\text{chain Rule}} = \frac{d(x^3)}{dx} + \frac{d}{dx}(y^3)$$

$$3(x+y)^2\left[\frac{d}{dx}(x) + \frac{d}{dx}(y)\right] = 3x^2 + 3y^2 \cdot \frac{dy}{dx}$$

$$3(x+y)^2 \cdot \left[1 + \frac{dy}{dx}\right] = 3x^2 + 3y^2 \cdot \frac{dy}{dx}$$

$$3(x+y)^2 + 3(x+y)^2 \cdot \frac{dy}{dx} = 3x^2 + 3y^2 \cdot \frac{dy}{dx}$$

$$3(x+y)^2 - 3x^2 = 3y^2 \cdot \frac{dy}{dx} - 3(x+y)^2 \cdot \frac{dy}{dx}$$

$$3(x+y)^2 - 3x^2 = \frac{dy}{dx}\left[3y^2 - 3(x+y)^2\right]$$

$$\frac{3(x+y)^2 - 3x^2}{3y^2 - 3(x+y)^2} = \frac{dy}{dx}$$

$$\frac{3\cdot((x+y)^2 - x^2)}{3\cdot(y^2 - (x+y)^2)} = \frac{dy}{dx}$$

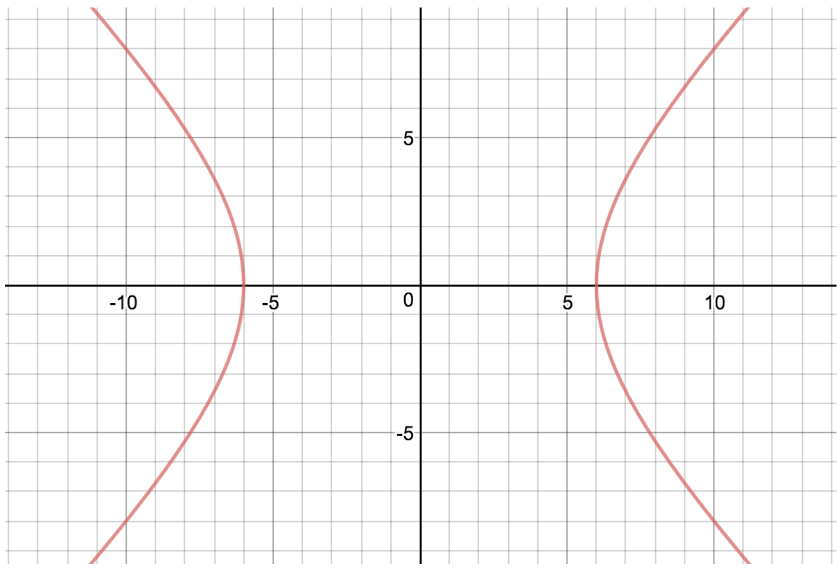
$$\boxed{\frac{(x+y)^2 - x^2}{y^2 - (x+y)^2} = \frac{dy}{dx}}$$

$$\left.\frac{dy}{dx}\right|_{(-1,1)} = \frac{(-1+1)^2 - (-1)^2}{(1)^2 - (-1+1)^2}$$

$$= \frac{-1}{1}$$

$$m_{tan}\Big|_{(-1,1)} = -1$$

Ex.6 Find $\dfrac{d^2y}{dx^2}$, given that $x^2 - y^2 = 36$.



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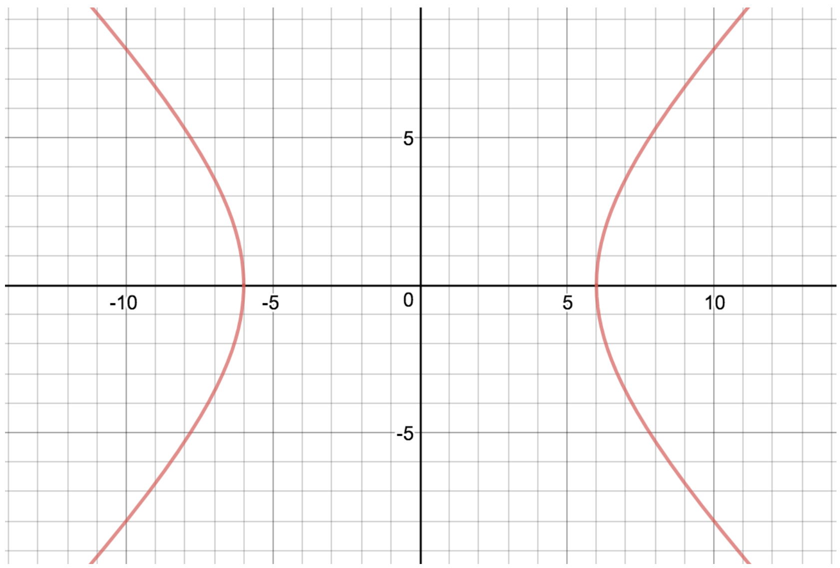

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